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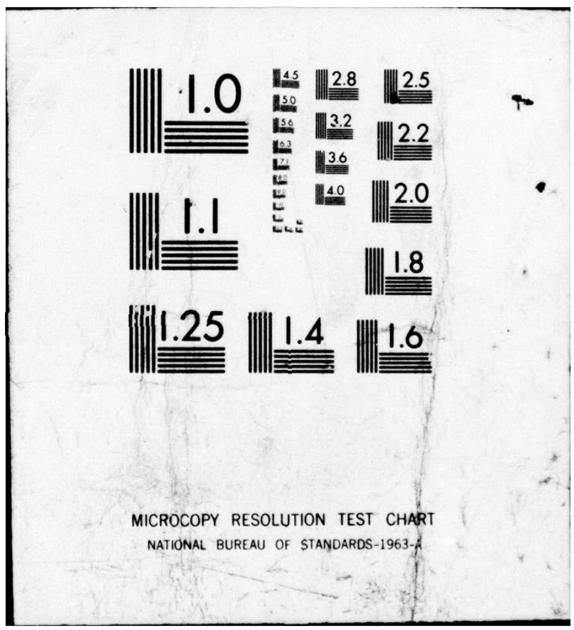
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THE STRONG UNIFORM CONSISTENCY OF NEAREST NEIGHBOR DENSITY ESTIMATES

Running Head: NEAREST NEIGHBOR DENSITY ESTIMATES

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THE STRONG UNIFORM CONSISTENCY OF

NEAREST NEIGHBOR DENSITY ESTIMATES

Summary. Let $X_1, \ldots, X_n$ be independent, identically distributed random vectors with values in $\mathbb{R}^d$ and with a common probability density f. If $V_k(x)$ is the volume of the smallest sphere centered at x and containing at least k of the $X_1, \ldots, X_n$ then $f_n(x) = k/(nV_k(x))$ is a nearest neighbor density estimate of f. We show that if $k = k(n)$ satisfies $k(n)/n \to 0$ and $k(n)/\log n \to \infty$ then $\sup_x |f_n(x) - f(x)| \to 0$ w.p.1 when f is uniformly continuous on $\mathbb{R}^d$.

Introduction.

Suppose that $X_1, \ldots, X_n$ are independent, identically distributed random vectors with values in $\mathbb{R}^d$ and with a common probability density f. If $V_k(x)$ is the volume of the smallest sphere centered at x and containing at least k of the random vectors $X_1, \ldots, X_n$, then Loftsgaarden and Quesenberry (1965), to estimate $f(x)$ from $X_1, \ldots, X_n$, let

$$f_n(x) = k/(nV_k(x)) \tag{1}$$

where $k = k(n)$ is a sequence of positive integers satisfying

$$\begin{aligned} &\text{(a)} \quad k(n) \uparrow \infty \\ &\text{(b)} \quad k(n)/n \to 0 \ . \end{aligned} \tag{2}$$

(The factor $k-1$ was used instead of k by Loftsgaarden and Quesenberry; this has no effect on any of the asymptotic results stated here.) They showed that $f_n(x)$ is a consistent estimate of $f(x)$ at each point where f is continuous and positive. This result can also easily be inferred from the work of Fix and Hodges (1951). For $d = 1$, Moore and Henrichon (1969) showed that

$$\sup_x |f_n(x) - f(x)| \to 0 \text{ in probability}$$

if f is uniformly continuous and positive on $\mathbb{R}$ and if, additionally,

$$k(n)/\log n \to \infty \ . \tag{3}$$

Wagner (1973) showed that $f_n(x)$ is a strongly consistent estimate of $f(x)$ at each continuity point of f if, in addition to (2b),

$$\sum_1^{\infty} e^{-\alpha k(n)} < \infty \text{ for all } \alpha > 0 \ . \tag{4}$$

(Notice that (4) is always implied by (3) but (2a) and (3) are needed to imply (4).) The result of this paper is the following theorem.

Theorem. If f is uniformly continuous on $\mathbb{R}^d$ and if k(n) satisfies (2b) and (3) then

$$\sup_x |f_n(x) - f(x)| \overset{n}{\to} 0 \text{ w.p.1.}$$

If

$$\hat{f}_n(x) = \sum_{i=1}^{n} K((x-X_i)/r(n))/nr(n)^d ,$$

where K is the uniform probability density for the unit sphere in $\mathbb{R}^d$ and $\{r(n)\}$ is a sequence of positive numbers, the recent results of Moore and Yackel (1977) (see Theorem 3.1) and the above theorem immediately yield that

$$\sup_x |\hat{f}_n(x) - f(x)| \to 0 \text{ w.p.1}$$

whenever f is uniformly continuous on $\mathbb{R}^d$ and $r(n) \to 0$, $nr(n)^d/\log n \to \infty$. This fact, an improvement over the previously published convergence results for the kernel estimate with a uniform kernel (e.g., see Theorem 2.1 of Moore and Yackel (1977)), also is a special case of Theorem 4.9 of Devroye (1976) who proves the same statement for all kernels K which are bounded probability densities with compact support and whose discontinuity points have a closure with Lebesgue measure 0.

Proof.

To simplify notation we assume below that multiplications are always carried out before division. Let $\epsilon > 0$ and choose $\delta > 0$ such that

$$|f(y) - f(x)| < \epsilon/2$$

whenever x and y are within a sphere of volume δ. Deferring measurability arguments for the moment,

$$P\{\sup_x |f_n(x) - f(x)| > \epsilon\} =$$

$$P\{\bigcup_x [V_k(x) < k/n(f(x) + \epsilon)]\} +$$

$$P\{\bigcup_{x: f(x) > \epsilon} [V_k(x) > k/n(f(x) - \epsilon)]\} .$$

The event $\bigcup_x [V_k(x) < k/n(f(x) + \epsilon)]$ implies that, for some x, there must be a sphere centered at x with volume less than $k/n(f(x) + \epsilon)$ and containing k of the random vectors $X_1, \ldots, X_n$. If $k/n\epsilon < \delta$ then the probability measure of such a sphere must be less than $\frac{k(f(x) + \epsilon/2)}{n(f(x) + \epsilon)}$ so that, for one of these spheres S,

$$\mu_n(S) - \mu(S) > \frac{k}{n} - \frac{k(f(x) + \epsilon/2)}{n(f(x) + \epsilon)}$$

$$= \frac{k\epsilon}{2n(f(x) + \epsilon)} \geq \frac{k\epsilon}{2n(F + \epsilon)}$$

where F is the maximum of f on $\mathbb{R}^d$, μ is the measure on the Borel subsets of $\mathbb{R}^d$ corresponding to f and μ_n is the empirical measure on the Borel subsets of $\mathbb{R}^d$ for $X_1, \ldots, X_n$. Thus, for $k/n\epsilon < \delta$,

$$P\{\bigcup_x [V_k(x) < k/n(f(x) + \epsilon)]\} \leq$$

$$P\{\sup_{S \in \mathcal{G}_n} |\mu_n(S) - \mu(S)| > k\epsilon/2n(F + \epsilon)\} \tag{5}$$

where $\mathcal{G}_n$ is the class of all spheres in $\mathbb{R}^d$ whose volume is less than $4k/n\epsilon$. Next, with $4k/n\epsilon < \delta$,

$$\bigcup_{x:f(x)>\epsilon} [V_k(x) > k/n(f(x)-\epsilon)] \subseteq$$

$$\bigcup_{x:f(x)>\epsilon} [V_k(x) > k/n(f(x) - (3\epsilon/4))]$$

which implies that, for some x with $f(x) > \epsilon$, there is a sphere S centered at x, with volume $\leq 4k/n\epsilon$, and

$$\mu(S) \geq k(f(x) - \epsilon/2)/n(f(x) - (3/4)\epsilon),$$

$$\mu_n(S) \leq k/n, \text{ and}$$

$$\mu(S) - \mu_n(S) \geq k\epsilon/4n(f(x) - (3/4)\epsilon) .$$

Thus

$$P\{ \bigcup_{x:f(x)>\epsilon} [V_k(x) > k/n(f(x) - \epsilon)] \} \leq \tag{6}$$
$$P\{ \sup_{S\in\mathcal{G}_n} |\mu(S) - \mu_n(S)| \geq k\epsilon/4nF \} ,$$

so that

$$P\{\sup_x |f_n(x) - f(x)| \geq \epsilon\} \leq 2P\{ \sup_{S\in\mathcal{G}_n} |\mu_n(S) - \mu(S)| \geq k\epsilon/4n(F+\epsilon) \} .$$

The proof will be completed if we show that for each $\epsilon > 0$

$$\sum_n P\{ \sup_{S\in\mathcal{G}_n} |\mu_n(S) - \mu(S)| \geq k\epsilon/4n(F+\epsilon) \} < \infty . \tag{7}$$

To prove (7) we employ a variation of the argument used by Vapnik and Chervonenkis (1971). In this variation use will be made of the following result. If $Y_1, \ldots, Y_n$ represent independent drawings without replacement from a population of k 0's and 1's then, for $\epsilon > 0$ and $k \geq n$,

$$P\left[\left|\left(\sum_{1}^{n} Y_i\right)/n - \mu\right| \geq \epsilon\right] \leq 2e^{-n\epsilon^2/(2\mu + \epsilon)} \tag{8}$$

where μ, the {number of 1's}/k, is assumed to be $\leq \frac{1}{2}$. Additionally (8) holds when $Y_1, \ldots, Y_n$ are Bernoulli random variables with parameter $\mu \leq \frac{1}{2}$. (Use the two-sided version of Theorem 3 of Hoeffding (1963) along with $\mu \leq \frac{1}{2}$ and $\log(1 + (\epsilon/\mu)) \geq 2\epsilon/(2\mu + \epsilon)$. See also section 6 of this paper.)

Now, if $\sup_{\mathcal{G}} \mu(\mathcal{G}) \leq M$ and $n \geq 8M/\delta^2$, an easy modification of Lemma 1 of Vapnik and Chervonenkis (1971) yields

$$\begin{aligned} &P[\sup_{\mathcal{G}} |\mu_n(A) - \mu(A)| \geq \delta] \leq \\ &2P[\sup_{\mathcal{G}} |\mu_n(A) - \mu_n'(A)| \geq \delta/2] \end{aligned} \tag{9}$$

where $\mu_n'(A)$ is the empirical measure for A with $X_{n+1}, \ldots, X_{2n}$ and $\mathcal{G}$ is any class of Borel sets in $\mathbb{R}^d$ for which

$$\sup_{\mathcal{G}} |\mu_n(A) - \mu(A)| \quad \text{and}$$

$$\sup_{\mathcal{G}} |\mu_n(A) - \mu_n'(A)|$$

are random variables. Putting $\mathcal{G} = \mathcal{G}_n$ we see that M can be taken to be $4kF/n\epsilon$. Since, for $\alpha > 0$,

$$\begin{aligned} &P[\sup_{\mathcal{G}_n} |\mu_n(A) - \mu'_n(A)| \geq \delta/2] \leq \\ &P[\sup_{\mathcal{G}_n} |\mu_n(A) - \mu_n'(A)| \geq \delta/2 \ ; \ \sup_{\mathcal{G}_n} \mu_{2n}(A) \leq \alpha M] \\ &+ P[\sup_{\mathcal{G}_n} \mu_{2n}(A) > \alpha M] \end{aligned} \tag{10}$$

we see, using (3) and putting $\delta = k\epsilon/4n(F+\epsilon)$, that (7) follows whenever both terms of the right-hand side of (10) are summable for some $\alpha > 0$. Looking at the first term, we note that it equals

$$\int_{\mathbb{R}^{2nd}} \frac{1}{(2n)!} \sum I_{[\sup_{\mathcal{G}_n} |\mu_n(A) - \mu_n'(A)| \geq \delta/2]} I_{[\sup_{\mathcal{G}_n} \mu_{2n}(A) \leq \alpha M]} dQ$$

where I_E is the indicator of the set $E \subseteq \mathbb{R}^d$ and Q is the probability measure on $\mathbb{R}^{2nd}$ for $X_1, \ldots, X_{2n}$ and where the inner summation is taken over all (2n)! permutations of $x_1, \ldots, x_{2n}$. But this last integral equals

$$\int_{\mathbb{R}^{2nd}} \frac{1}{(2n)!} \sum I_{[\sup_{\mathcal{G}_n} \mu_{2n}(A) \leq \alpha M]} \sup_{\mathcal{G}_n} I_{[|\mu_n(A) - \mu_n'(A)| \geq \delta/2]} dQ$$

$$= \int_{\mathbb{R}^{2nd}} \frac{1}{(2n)!} \sum I_{[\sup_{\mathcal{G}_n} \mu_{2n}(A) \leq \alpha M]} \sup_{\mathcal{G}'} I_{[|\mu_n(A) - \mu_n'(A)| \geq \delta/2]} dQ$$

$$\leq \int_{\mathbb{R}^{2nd}} \sum_{A \in \mathcal{G}'} I_{[\sup_{\mathcal{G}_n} \mu_{2n}(A) \leq \alpha M]} \left\{ \frac{1}{(2n)!} \sum I_{[|\mu_n(A) - \mu_{2n}(A)| \geq \delta/4]} \right\} dQ$$

where $\mathcal{G}' = \mathcal{G}'(x_1, \ldots, x_{2n})$ is any finite subclass of $\mathcal{G}_n$ which yields the same class of intersections with $\{x_1, \ldots, x_{2n}\}$ and where the inner summation is again taken over the (2n)! permutations of $x_1, \ldots, x_{2n}$. The quantity within $\{\cdot\}$ is bounded above, using (8), by

$$2e^{-n\delta^2/(32\mu_{2n}(A) + 4\delta)}$$

whenever $\mu_{2n}(A) \leq \frac{1}{2}$. Since $M = 4kF/n\epsilon$ we see, from (3), that for all n sufficiently large the last integral is upper-bounded by

$$2\int_{\mathbb{R}^{2nd}} e^{-n\delta^2/(32\alpha M+4\delta)}\left(\sum_{A\in G'} 1\right) dQ\ .$$

Choosing G' to be a smallest possible subclass, we have (Vapnik and Chervonenkis (1971), Cover (1965)) that $\left(\sum_{A\in G'} 1\right) \leq 1 + (2n)^{d+3}$ and, using (3) again, that the first term of (10) is summable for all $\alpha > 0$.

Looking at the second term of (10), let r be the radius of a sphere in $\mathbb{R}^d$ whose volume is $4k/n\epsilon$. If some sphere of radius r contains ℓ of the points $X_1, \ldots, X_{2n}$ then there must be at least one sphere of radius 2r, centered at one of the points $X_1, \ldots, X_{2n}$, which contains at least ℓ points. Thus

$$P[\sup_{G_n} \mu_{2n}(A) > \alpha M] \leq 2nP[\mu_{2n}(S_{X_1}(2r)) > \alpha M]$$

where $S_x(t)$ denotes the sphere of radius t centered at x. But

$$P[\mu_{2n}(S_{X_1}(2r)) > \alpha M] \leq$$

$$\max_{x\in\mathbb{R}^d} P[\mu_{2n-1}(S_x(2r)) > (\alpha 2nM-1)/(2n-1)]$$

$$\leq \max_{x\in\mathbb{R}^d} P[\mu_{2n-1}(S_x(2r)) > [(\alpha 2nM-1)/(2n-1)] - 2^d 4kF/n\epsilon]\ .$$

At this point it is not difficult, using (3) and (8), to show that the second term of (9) is summable as long as $\alpha > 2^d$.

Finally, to complete the proof, it is easy to see that all of the uncountable unions over x are indeed events and that the various supremums over G_n are indeed random variables.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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20. Abstract (continued)

We show that if $k = k(n)$ satisfies $k(n)/n \to 0$ and $k(n)/\log n \to \infty$ then $\sup_x |f_n(x) - f(x)| \to 0$ w.p.1 when f is uniformly continuous on $\mathbb{R}^d$.

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